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A PERTURBATION METHOD FOR MIXED THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY WITH A COMPLEX LINE OF BOUNDARY-CONDITION SEPARATION*

N.M. BORODACHEV

A modification of the perturbation method is proposed, based on the utilization of variational formulas and enabling asymptotic expansions (AE) to be obtained for mixed three-dimensional problems of the theory of elasticity with a complex line of boundary-condition separation. Application of Lighthill's method enables these expansions to be transformed into uniformly suitable ones. The problem for an elastic body with a slit (crack) and the contact problem of the theory of elasticity are considered separately. For the body with a slit the variational formula determines the variation of the displacement of the slit surface caused by variation in the shape of the slit contour. The effectiveness of this formula for constructing AE in problems associated with a perturbation of the shape of the slit contour is shown. Cases of slits of complex shape in an infinite body that differ slightly from a circular slit are examined in detail. A scheme for constructing similar AE is mentioned for spatial contact problems of the theory of elasticity with a complex shape of the contact area.

A review of the application of perturbation methods to mixed problems in the theory of elasticity is contained in /1, 2/. The solutions of mixed spatial problems in the theory of elasticity with a complex line of boundary condition separation, obtained by using other methods, are discussed in /3-8/. The behaviour of the solution of the boundary value problem for a pseudodifferential equation (in particular, crack theory) for variation of the domain was investigated in /9/.

1. We consider a linearly elastic body occupying a simply-connected volume V. Let O be the surface bounding this volume. There is a plane slit of surface S in the body. A kinematic boundary condition is given on the part O_1 of the body surface and a static condition on its other part O_2. The boundary contour of the slit Γ is a plane curve. We use a rectangular system of coordinates x_1, x_2, x_3. The slit is in the plane $x_3 = 0$. We associate the positive orientation S^+ of the surface S with the limit value $x_3 = 0^+$ and the negative orientation S^- with $x_3 = 0^-$. The slit surfaces S^+ and S^- are contained in O_2, i.e., a static boundary condition is given on the slit surface.

Let us magnify the size of the slit by displacing the contour Γ in a nearby location Γ_1. At each point $M \in \Gamma$ we direct the variation $\delta n(M)$ along the outer normal to the curve Γ.

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The system potential energy Φ in this case is a functional over $\mathbf{u}$ and Γ, i.e., $\Phi = \Phi(\mathbf{u}, \Gamma)$, where $\mathbf{u}$ is the displacement vector. Variation of the slit contour δn will cause a variation of the functional Φ which we will denote by $\delta_n\Phi$. It can be shown that

$$\delta_n\Phi = 0 \tag{1.1}$$

We will confine ourselves to considering the normal separation case. Then, written in detail, the stationarity condition (1.1) takes the form

$$\int_\Gamma K_1^2(M)\,\delta n(M)\,ds = -\alpha \iint_{S^+} \sigma_{33}(Q)\,\delta_n u_3(Q)\,dS \tag{1.2}$$

$$\alpha = 2\mu/(\pi(1-\nu)),\quad M \in \Gamma,\ Q \in S^+$$

where μ is the shear modulus, ν is Poisson's ratio and K_1 is the stress intensity factor of normal separation. It is assumed here that the load is applied only to the slit surfaces S^+ and S^-. Identical normal stresses $\sigma_{33}(Q)$ and tangential stresees $\sigma_{13}(Q) = \sigma_{23}(Q) = 0$ are given on both surfaces S^+ and S^-.

We will examine two states of equilibrium of this body with a slit, later called first and second. In the first state, unit concentrated forces $\sigma_{33}^{(1)}(Q) = -\delta(Q, Q_1)$, are applied to the surfaces S^+ and S^-, and a certain pressure $p(Q)$ in the second state, i.e., $\sigma_{33}^{(2)}(Q) = -p(Q)$. Here $\delta(Q, Q_1)$ is a delta function. Moreover, we consider the total state for which

$$\sigma_{33}(Q) = \sigma_{33}^{(1)}(Q) + \sigma_{33}^{(2)}(Q)$$

Applying (1.2) to these states and using the theorem on reciprocity of work /10/, we obtain

$$\delta_n u_3^{(2)}(Q) = \alpha^{-1} \int_\Gamma K_1^{(1)}(M;Q)\,K_1^{(2)}(M)\,\delta n(M)\,ds \tag{1.3}$$

Here $K_1^{(1)}$ corresponds to $\sigma_{33}^{(1)}$ while $K_1^{(2)}$ and $u_3^{(2)}$ correspond to $\sigma_{33}^{(2)}$. Formula (1.3) expresses the variation of the displacement of the slit surface S^+ caused by variation of the slit contour. It is assumed here that a pressure $p(Q)$ is given on the slit surfaces.

We will examine a special case of (1.3). Let Γ be a circle of radius a described around the origin. By using a cylindrical coordinate system r, θ, z (1.3) takes the form

$$\delta_n u_z^{(2)}(r,\theta) = \frac{\pi(1-\nu)a}{2\mu} \int_0^{2\pi} K_1^{(1)}(\varphi; r, \theta)\,K_1^{(2)}(\varphi)\,\delta n(\varphi)\,d\varphi \tag{1.4}$$

where φ is the polar angle corresponding to the point M. Furthermore, let the slit be in an infinite body. Then we have for a plane circular slit /11/

$$K_1^{(1)}(\varphi; r, \theta) = \frac{(a^2-r^2)^{1/2}}{\pi^2 a^{1/2}[a^2+r^2-2ar\cos(\varphi-\theta)]}$$

Substituting this expression into (1.4) and discarding the superscripts indicating the number of the state, we finally obtain

$$\delta_n u_z(r,\theta) = \frac{(1-\nu)a^{1/2}(a^2-r^2)^{1/2}}{2\pi\mu} \int_0^{2\pi} \frac{K_1(\varphi)\,\delta n(\varphi)\,d\varphi}{a^2+r^2-2ar\cos(\varphi-\theta)} \quad (r < a) \tag{1.5}$$

Formula (1.5) enables us to determine the variation of the displacement of the plane slit surface S^+ in an infinite elastic body during passage of the slit contour Γ (Γ is a circle of radius a) into a nearby location Γ_1. Variation of the slit contour is determined by the quantity $\delta n(\varphi)$. It is assumed that an arbitrary normal load $\sigma_z(r,\theta) = -p(r,\theta)$ is applied to the slit surfaces, for which the stress intensity factor is $K_1(\varphi)$, which is determined for the contour Γ.

The variational formulas (1.3)-(1.5) obtained can be used when solving different problems for bodies with a slit. In particular, these formulas turned out to be effective in constructing the AE in problems associated with a perturbation in the shape of the slit contour.

2. Let there be a plane slit of complex shape in an infinite elastic body. The boundary contour of the slit Γ_1 differs slightly from a circle of radius a (the contour Γ) and its equation in polar coordinates has the form

$$\rho = a[1 + \varepsilon f(\varphi)],\quad \varepsilon \ll 1 \tag{2.1}$$

where $f(\varphi)$ is a certain piecewise-continuous function. Therefore, just a small perturbation of the contour of a circular slit is considered. We shall seek the solution of the perturbed problem in the form of an AE in the small parameter ε

$$u_z(r,\theta) = u_{z0}(r,\theta) + \varepsilon u_{z1}(r,\theta) + O(\varepsilon^2) \tag{2.2}$$

Here $u_{z0}(r, \theta)$ is the solution of the unperturbed problem (for a circular slit). The displacement $u_{z0}(r, \theta)$ can be found by using the method proposed in /12/.

To find the small correction εu_{z1} to the quantity u_{z0} it is possible to use (1.5). It follows from (2.1) that

$$\delta n(\varphi) = \varepsilon a f(\varphi), \quad \varepsilon \ll 1 \tag{2.3}$$

Since $\delta_n u_z = \varepsilon u_{z1}$, then by using (1.5) and (2.3) we will have

$$u_{z1}(r, \theta) = \frac{(1-\nu) a^{3/2}}{\mu (a^2 - r^2)^{1/2}} U(r, \theta) \tag{2.4}$$

$$U(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} K_{10}(\varphi) f(\varphi) \frac{(a^2 - r^2)\, d\varphi}{a^2 + r^2 - 2ar\cos(\varphi - \theta)} \quad (r < a)$$

where $K_{10}(\varphi)$ is calculated from the pressure $p(r, \theta)$ applied to the circular slit surface.

The Poisson integral $U(r, \theta)$ yields a harmonic function within the circle $r < a$ for an arbitrary piecewise-continuous function $K_{10}(\varphi) f(\varphi)$. The function $U(r, \theta)$ is bounded for $r \leqslant a$ and continuously joins the boundary values at the points of continuity of the function $K_{10}(\varphi) f(\varphi)$. Ordinarily $K_{10}(\varphi)$ is a continuous function and therefore, $f(\varphi)$ can be a piecewise-continuous function. Using the properties of the Poisson integral, a number of estimates can be obtained for the function $U(r, \theta)$ and, therefore, also for $u_{z1}(r, \theta)$.

Therefore, a formal AE (2.2) is constructed. Let us analyse it.

As $r \to a - 0$ the function $u_{z0}(r, \theta)$ behaves as $O((a - r)^{1/2})$ while the function $u_{z1}(r, \theta)$ behaves as $O((a - r)^{-1/2})$. The assumption of the smallness of the perturbation is violated near the critical point $r = a$, where the solution obtained is not uniformly suitable. By applying Lighthill's method, the uniform **suitability** of the AE (2.2) can be restored. We shall use the easily modified version of this method proposed in /13/.

3. We will examine specific examples of constructing uniformly suitable AE.

Let a uniform pressure, i.e.,

$$p(r, \theta) = p = \mathrm{const} \tag{3.1}$$

be applied to the slit surfaces.

As is well-known /11/, in this case

$$K_{10}(\varphi) = \frac{2pa^{1/2}}{\pi}, \quad u_{z0}(r, \theta) = \frac{2(1-\nu)p}{\pi\mu}(a^2 - r^2)^{1/2} \tag{3.2}$$

Substituting (3.2) into (2.2) and (2.4) we obtain

$$u_z(r, \theta) = \frac{2(1-\nu)p}{\pi\mu}\left[(a^2 - r^2)^{1/2} + \varepsilon \frac{a^2 F(r, \theta)}{(a^2 - r^2)^{1/2}}\right] + O(\varepsilon^2) \tag{3.3}$$

$$F(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\varphi)(a^2 - r^2)\, d\varphi}{a^2 + r^2 - 2ar\cos(\varphi - \theta)}$$

The perturbed displacement determined by (3.3) has a singularity as $r \to a$. Therefore this solution is not uniformly suitable near the critical point $r = a$.

The source of the inhomogeneity can be eliminated by using Lighthill's method. We replace r by a slightly deformed coordinate r_0. We set

$$r = r_0 + \varepsilon \Psi(r_0, \theta) \tag{3.4}$$

and we substitute it into the first formula in (3.3). The first-order approximation will obviously have a singularity of the same order as the zero-th approximation if $\Psi(r_0, \theta) = r_0 F(r_0, \theta)$. Then the expression in the square brackets in (3.3) equals

$$[1 + \varepsilon F(r_0, \theta)](a^2 - r_0^2)^{1/2}$$

Returing to the variable r, we finally obtain the uniformly suitable expansion

$$u_z(r, \theta) = \frac{2(1-\nu)p}{\pi\mu}\{[1 + 2\varepsilon F(r, \theta)]a^2 - r^2\}^{1/2} + O(\varepsilon^2) \tag{3.5}$$

Formula (3.5) determines the displacement of the slit surface S^+ whose boundary contour Γ_1 is given by (2.1). A plane slit exists in an infinite elastic body; a uniform pressure p is applied to the slit surfaces. The function $F(r, \theta)$ in (3.5) and which depends on the shape of the slit contour Γ_1 can be found by means of (3.3).

To verify (3.5) we will examine the standard problem for which an exact solution is known. Let the contour Γ_1 be an ellipse with semi-axes $(1 + \varepsilon)a$ and a. In this case (2.1) remains valid if $f(\varphi) = \cos^2\varphi$. Substituting this expression into (3.3) we find

$$F(r, \theta) = (a^2 + r^2\cos 2\theta)/(2a^2) \tag{3.6}$$

Going over to rectangular coordinates in (3.5) we obtain

$$u_z'(x_1, x_2) = \frac{2(1-\nu)pa}{\pi\mu}\left(1+\frac{\varepsilon}{2}\right)\left\{1-\frac{x_1^2}{[(1+\varepsilon)a]^2}-\frac{x_2^2}{a^2}\right\}^{1/2}+O(\varepsilon^2)$$

The same result is obtained from the known exact solution /11/.

We will now examine the more complex problem for which an exact solution is unknown. Let the slit boundary contour be given by (2.1) in which

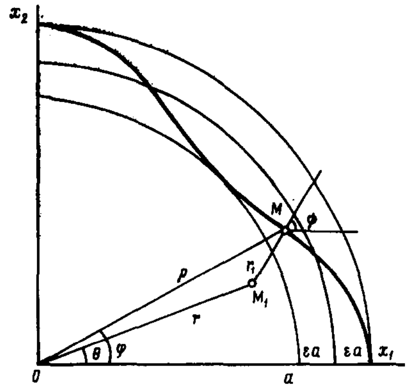
$$f(\varphi) = 1 + \cos n\varphi \tag{3.7}$$

n is an even positive number (the slit contour under consideration is shown for $n = 4$ for the first quadrant in the sketch since it is symmetrical about the x_1 and x_2 axes).

Substituting (3.7) into the second formula of (3.3) and integrating, we find

$$F(r, \theta) = 1 + (r/a)^n \cos n\varphi \tag{3.8}$$

Therefore (3.5) and (3.8) determine the displacement of the slit surface S^+ when its boundary contour is given by (2.1) and (3.7) and a uniform pressure is applied to the slit surfaces.



The expression for the stress intensity factor is of the greatest practical interest. To obtain this expression it is first necessary to find the asymptotic representation for u_z near the boundary contour of the slit. The coordinates r, θ of the point M_1 located near the slit boundary on the inner normal to the slit contour at the point M (sketch) with coordinates ρ, φ are needed here. It can be shown that

$$\operatorname{tg}\psi = (1 + 2\varepsilon n \sin n\varphi/\sin 2\varphi) \operatorname{tg}\varphi + O(\varepsilon^2) \tag{3.9}$$

$$r = (\rho - r_1) + O(\varepsilon^2), \quad \theta = \varphi + \varepsilon n a^{-1} r_1 \sin n\varphi + O(\varepsilon^2),$$

$$r_1 \to 0$$

Here r_1 is the spacing between the points M and M_1. Using (3.5), (3.8) and (3.9), we find the asymptotic representation for the displacement and the stress intensity factor

$$u_z \approx 2(1-\nu)pa^{1/2}(\pi\mu)^{-1}\{1 + {}^1/_2\varepsilon[1 + (1-n)\cos n\varphi] + O(\varepsilon^2)\}(2r_1)^{1/2}, \quad r_1 \to 0 \tag{3.10}$$

$$K_1 = 2\pi^{-1}pa^{1/2}\{1 + {}^1/_2\varepsilon[1 + (1-n)\cos n\varphi]\} + O(\varepsilon^2) \tag{3.11}$$

It follows from (3.11) for the slit contour shown in the sketch, i.e., for $n = 4$, that in the first quadrant (sketch) K_1 attains the greatest value for $\varphi = \pi/4$, and the least for $\varphi = 0$ and $\varphi = \pi/2$.

In the general case, it is seen from (3.11) that the greatest value of K_1 is directly proportional to $(1 + {}^1/_2\varepsilon n)$. Therefore, $K_{1\max}$ grows as the curvature of the slit boundary contour increases. However, the number n in (3.11) cannot be very large; its magnitude is constrained by the requirement that the correction ${}^1/_2\varepsilon n$ be small compared with one.

4. We will now consider the case when concentrated forces are applied to the slit surfaces. Let

$$p(r, \theta) = \frac{P}{2\pi r}\delta(r) \tag{4.1}$$

where $\delta(r)$ is the delta function. In this case

$$K_{10}(\varphi) = \frac{P}{\pi^2 a^{3/2}}, \quad u_{z0}(r, \theta) = \frac{P(1-\nu)}{\pi^2\mu r}\arccos\frac{r}{a} \tag{4.2}$$

On the basis of (2.2), (2.4) and (4.2) we have

$$u_z(r, \theta) = \frac{P(1-\nu)}{\pi^2\mu}\left[\frac{1}{r}\arccos\frac{r}{a} + \varepsilon\frac{F(r, \theta)}{(a^2 - r^2)^{1/2}} + O(\varepsilon^2)\right] \tag{4.3}$$

where $F(r, \theta)$ is determined by the second formula in (3.3). The solution (4.3) is not uniformly suitable. Consequently, we again use the change of variables (3.4) and Lighthill's method. We consequently find

$$r = r_0 [1 + \varepsilon F(r_0, \theta)] \qquad (4.4)$$
$$u_z(r, \theta) = \frac{P(1-\nu)}{\pi^2 \mu r} \arccos \frac{r}{[1 + \varepsilon F(r, \theta)] a} + O(\varepsilon^2)$$

The last expression is uniformly suitable. The singularity at $r = 0$ in the formula for u_z is caused by the nature of the load applied to the slit surfaces. As $r \to 0$ the second formula in (4.4) **yields** a result corresponding to the action of a concentrated force on the boundary of an elastic half-space.

Formula (4.4) for u_z is applicable to a plane slit whose boundary contour Γ_1 is given by (2.1).

As an illustration, let us consider the case when the boundary contour of the slit is an ellipse with the semi-axes $(1+\varepsilon)a$ and a. For such a slit the function $F(r, \theta)$ is determined by (3.6).

To determine the displacement u_z near the boundary of the elliptical slit, we will change from the coordinates x_1, x_2, x_3 to the coordinates r_1, θ_1, ψ /14/. For $\theta_1 = \pi$ we have

$$x_1 = \cos\psi\,(a_1 - a_2 r_1 \Pi_0) \qquad (4.5)$$
$$x_2 = \sin\psi\,(a_2 - a_1 r_1 \Pi_0)$$
$$\Pi_0 = (a_1^2 \sin^2\psi + a_2^2 \cos^2\psi)^{-1/2}$$
$$K_1 = \frac{\mu}{1-\nu} \lim_{r_1 \to 0} \frac{u_z}{(2r_1)^{1/2}}$$

(ψ is the parametric angle of the ellipse). In the case under consideration

$$a_1 = (1+\varepsilon)a, \quad a_2 = a$$
$$\sin\psi = \sin\varphi\,(1 + \varepsilon\cos^2\varphi) + O(\varepsilon^2)$$

where φ is the polar angle in (2.1). Using (4.4), (3.6) and (4.5) we find

$$u_z \approx \frac{P(1-\nu)}{\pi^2 \mu a^{3/2}} \left[1 + \frac{\varepsilon}{2}(\sin^2\psi - 4\cos^2\psi) + O(\varepsilon^2)\right](2r_1)^{1/2}, \qquad r_1 \to 0$$
$$K_1(\psi) = \frac{P}{\pi^2 a^{3/2}} \left[1 - \frac{\varepsilon}{4}(3 + 5\cos 2\psi)\right] + O(\varepsilon^2)$$

As the examples examined above show, (2.2) and (2.4) permit a **fairly** simple construction of AE for the displacement of a slit surface of complex shape in an infinite body when distributed loads or concentrated forces are applied to the slit surfaces. Using Lighthill's method, this AE can be made uniformly suitable and a formula can be obtained to evaluate the stress intensity factor.

5. We will examine the spatial contact problem of the theory of elasticity. Let a linearly elastic body occupy the simply-connected volume V. The surface O bounding this volume consists of a certain surface O_1 and the plane surface O_0 whose equation is $x_3 = 0$. A rigid cylindrical stamp of arbitrary cross-section is impressed into the plane surface of the body O_0. The stamp base has the shape of a convex surface. The plane surface O_0 is partitioned into two parts, the contact area O_3 and the surface O_2 (O_2 is considered stress free). A static boundary condition is given on O_1.

In this case the contour of the contact area Γ together with the stress tensor are inserted into a number of independent elements characterizing the state of the elastic body. The equation /15/

$$\int_\Gamma K_1^2(M)\,\delta n(M)\,ds = -\frac{2\mu}{\pi(1-\nu)} \iint_{O_3} u_3(Q)\,\delta_n \sigma_{33}(Q)\,dS \qquad (5.1)$$
$$M \in \Gamma, \quad Q \in O_3$$

is obtained from the variation of the contact area contour by using the functional of the principle of the minimum of additional work, where K_1 is the compressive stress **intensity** factor, δn is the variation of the contact area contour, and $\delta_n \sigma_{33}$ is the variation of the stress σ_{33} on the contact area caused by the variation δn.

Formula (5.1) plays the same role for the spatial contact problem of the theory of elasticity as does (1.2) for a body with a slit. Consequently, further solution of the contact problem with a complex shape of the contact area can be carried out by the same scheme as for the problem with a complex shape of a plane slit.

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THE NON-LINEAR DYNAMICS OF ELASTIC RODS*

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The general equations of non-linear dynamics of elastic rods are examined taking tension, transverse shear, eccentricity, rotational inertia, and also initial stresses into account. A second-order theory is constructed for Timoshenko and classical-type models. A variational formulation is given for the linearized problem. Tension and shear effects are examined in the problem of the stability of a compressed column.

1. Geometry and kinematics. A rod is considered below to be a deformable material line whose particles are solids /1/. A Lagrange coordinate s, $0 \leqslant s \leqslant l$ is introduced. This usually an arc coordinate in a reference configuration. The rod motion is determined by the time dependence of the radius-vector $\mathbf{r}(s, t)$ and the rotation tensor $\mathbf{P}(s, t)$ for each particle. Internal interactions are given by the force vector $\mathbf{Q}(s, t)$ and moment vector $\mathbf{M}(s, t)$ with which a particle with coordinate $s+0$ acts on a neighbour $s-0$ ($\mathbf{Q}$ and $\mathbf{M}$ change when the reference direction s is reversed).

To assign an angular orientation, an orthogonal triple $\mathbf{e}_k$ is associated with each particle according to a certain rule; it is often assumed, say, that $\mathbf{e}_{30} = \mathbf{r}_0'$ ($(\ldots)' = \partial/\partial s$; the zero subscript marks quantities in the reference configuration). By the definition of the rotation tensor $\mathbf{e}_k = \mathbf{P}\cdot\mathbf{e}_{k0}$, $\mathbf{P} = \mathbf{e}_k\mathbf{e}_{k0}$. Here and henceforth, the language of the direct tensor calculus is used /2/. The curvature vector and rod twist are introduced by the relationships $\mathbf{e}_k' = \boldsymbol{\Omega} \times \mathbf{e}_k$, $\boldsymbol{\Omega} = {}^1/_2\mathbf{e}_k \times \mathbf{e}_k'$ As will be shown below, the vectors

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